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Two-body relaxation in modified Newtonian dynamics

Luca Ciotti[1] & James Binney[2]
[1] *Astronomy Department, University of Bologna, via Ranzani 1, 40127 Bologna, Italy*
[2] *Oxford University, Theoretical Physics, 1 Keble Road, Oxford OX1 3NP*

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ABSTRACT

A naive extension to MOND of the standard computation of the two-body relaxation time t_{2b} implies that t_{2b} is comparable to the crossing time regardless of the number N of stars in the system. This computation is questionable in view of the non-linearity of MOND's field equation. A non-standard approach to the calculation of t_{2b} is developed that can be extended to MOND whenever discreteness noise generates force fluctuations that are small compared to the mean-field force. It is shown that this approach yields standard Newtonian results for systems in which the mean density profile is either plane-parallel or spherical. In the plane-parallel case we find that in the deep-MOND regime t_{2b} scales with N as in the Newtonian case, but is shorter by the square of the factor by which MOND enhances the gravitational force over its Newtonian value for the same system. Near the centre of a spherical system that is in the deep-MOND regime, we show that the fluctuating component of the gravitational force is never small compared to the mean-field force; this conclusion surprisingly even applies to systems with a density cusp that keeps the mean-field force constant to arbitrarily small radius, and suggests that a cuspy centre can never be in the deep MOND regime. Application of these results to dwarf galaxies and groups and clusters of galaxies reveals that in MOND luminosity segregation should be far advanced in groups and clusters of galaxies, two body relaxation should have substantially modified the density profiles of galaxy groups, while objects with masses in excess of $\sim 10M_\odot$ should have spiralled to the centres of dwarf galaxies.

Key words: gravitation – galaxies:dwarf – galaxies: haloes – galaxies: kinematics and dynamics

1 INTRODUCTION

Milgrom (1983) proposed that the failure of galactic rotation curves to decline in Keplerian fashion outside the galaxies' luminous body arises not because galaxies are embedded in massive dark haloes, but because Newton's law of gravity has to be modified for fields that generate accelerations smaller than some value a_0. Bekenstein & Milgrom (1984; hereafter BM84) proposed the non-relativistic field equation, eq. (2) below, for the gravitational potential Φ that generates an appropriately modified gravitational acceleration $\mathbf{g}$ of a *test* particle through

$$\mathbf{g} = -\nabla\Phi. \tag{1}$$

A considerable body of observational data now supports this theory of modified Newtonian gravity (MOND) – see Sanders & McGaugh (2002) for a review. The big problem with MOND is our inability to derive the MOND field equation as the low-energy limit of a Lorentz covariant theory. This inability is unfortunate in two respects. First it makes it impossible to determine MOND's predictions for gravitational lensing experiments, or any observation that involves relativistic cosmology, particularly observations of the CMB. Since these are the areas in which the competing Cold Dark Matter (CDM) theory has been most successful, the lack of a Lorentz covariant form of MOND makes a fair confrontation between MOND and CDM impossible. Another reason to regret this lack is that there are tantalizing hints that the characteristic acceleration $a_0 \simeq 2\times10^{-8}$ cm s^{-2} that lies at the heart of MOND is connected to the requirement for a non-zero cosmological constant Λ in Einstein's equations: a_0 and $\Lambda \simeq 3(a_0/c)^2$ may be two aspects of a single physical process associated with the unknown small-scale structure of space-time (Milgrom 2002).

Clearly it is worthwhile to pursue these ideas regarding a putative Lorentz covariant form of MOND only if the low-energy theory that we already have accounts for all available data. In this paper we show that MOND predicts two-body relaxation times for systems in which $g \ll a_0$ that are shorter than those given by Newtonian theory by a

factor $\sim (g/a_0)^2$. We show further that in MOND dynamical friction operates on a timescale that is shorter than in Newtonian dynamics with dark matter by a factor $\sim g/a_0$. The shortness of the dynamical friction timescale has observationally testable predictions for the dynamics of systems such as dwarf galaxies, groups and clusters of galaxies that are in the deep MOND regime.

2 ORDER-OF-MAGNITUDE ANALYSIS

BM84 replace Poisson's equation for the potential in terms of the density ρ by

$$\nabla \cdot [\mu(|\nabla\Phi|/a_0)\nabla\Phi] = 4\pi G\rho, \tag{2}$$

to be solved subject to the boundary condition $|\nabla\phi| \to 0$ for $|\mathbf{x}| \to \infty$. The function $\mu(x)$ is required to have the behaviour

$$\mu \simeq \begin{cases} x & \text{for } x \ll 1, \\ 1 & \text{for } x \gg 1, \end{cases} \tag{3}$$

but the detailed manner in which μ moves between these limits is currently constrained by neither observational data nor theory.

We shall be concerned with the 'deep MOND regime' in which $\mu(x) \simeq x$. In this limit equation (2) can be written

$$\nabla \cdot \left[\frac{|\nabla\Phi|\nabla\Phi}{a_0} - \nabla\Phi_{\rm N}\right] = 0, \tag{4}$$

where $\Phi_{\rm N}$ is the Newtonian potential generated by the given density distribution. This equation implies that the difference between the two terms in the square brackets is equal to the curl of some vector field. BM84 show that when the density distribution is spherical, planar or cylindrical, the curl vanishes. It then follows that the acceleration $\mathbf{g}_0$ in MOND is related to the Newtonian acceleration $\mathbf{g}_{\rm N}$ by

$$g_0^2 = a_0 g_{\rm N}. \tag{5}$$

We now present a heuristic derivation of the two-body relaxation time of a system that is in this regime. Our analysis is modelled on the standard Newtonian derivation of the two-body time (e.g., §4.1 of Binney & Tremaine 1987, hereafter BT). This derivation is straightforward to follow and illuminates the basic physical principle that causes two-body relaxation to be fast in MOND. It is open to the criticism, however, that it ignores the inherent non-linearity of the basic equation (2). Consequently, in the following section we rederive the Newtonian relaxation time by a very different technique that carries effortlessly over to the case of MOND.

Milgrom (1986; hereafter M86) shows that in the deep MOND regime equation (2) causes a force F to act between two isolated point masses m_1 and m_2, which can be written

$$F = \frac{Gm_1m_2}{r^2} f\left(\frac{m_2}{m_1}, \frac{r}{r_0}\right), \tag{6}$$

where

$$r_0 \equiv \sqrt{\frac{G(m_1+m_2)}{a_0}} \simeq 8.1\,10^{16}\sqrt{\frac{m_1+m_2}{M_\odot}} \quad \text{cm}. \tag{7}$$

The function f in equation (6) can be calculated numerically as a function of r/r_0 for a given mass ratio. In particular, numerical/asymptotic solution of equation (2) shows that:

- for $r/r_0 \leq 1$, $f \sim 1$ independently of the mass ratio;
- for $m_2 \ll m_1$, the acceleration of m_2 is given by equation (1), while the acceleration of m_1 is obtained from the conservation of linear momentum $m_1\mathbf{g}_1 + m_2\mathbf{g}_2 = 0$;
- finally, for nearly equal masses in the *deep-MOND regime*,

$$F \simeq \frac{m_1m_2}{\sqrt{m_1+m_2}}\frac{\sqrt{Ga_0}}{r} = \frac{Gm_1m_2}{rr_0}. \tag{8}$$

According to the numerical calculations of M86, this formula does not err by more than 20% for any mass ratio. Note that its asymptotic behavior is correct when one of the masses is vanishingly small or much more massive than the other.

For ordinary stars, r_0 is much less than the mean interstellar distance in galaxies, so nearly all stellar interactions are in the deep-MOND regime if the local gravitational mean field g of the whole galaxy is. If, by contrast, $g \gtrsim a_0$, then all interactions, even ones that are individually weak, conform to Newtonian theory – this result is sometimes called the "external field effect" (BM84). An immediate consequence of this result is that MOND predicts standard two-body relaxation times for objects in which the mean field exceeds a_0, even though individual stellar interactions generate accelerations much smaller than a_0.

Consider then a stellar encounter in a system that is in the deep-MOND regime. For simplicity we assume that both stars have the same mass m. Let the encounter be characterized by impact parameter $b \geq r_0$ and asymptotic relative velocity $\mathbf{V}$. Using equation (8) and the impulse approximation, we conclude that the encounter changes the velocity of each star (in the direction perpendicular to $\mathbf{V}$) by an amount

$$\Delta v_\perp \simeq \frac{2b}{V} \times \frac{F(b)}{m} = \frac{2Gm}{r_0V}. \tag{9}$$

Remarkably, this formula does not contain the impact parameter b: *in the deep-MOND regime, the deflection is independent of the impact parameter.* The steady accumulation of such velocity changes causes the star's velocity to execute a random walk. If the system contains N stars and has half-mass radius R, per crossing time the star experiences

$$\delta n = \frac{N/2}{\pi R^2} 2\pi b \mathrm{d}b \tag{10}$$

encounters with impact parameter in $(b+\mathrm{d}b, b)$. Adding in quadrature the velocity changes from successive encounters on the assumption that they are uncorrelated yields a cumulative change in the square of the stellar speed per crossing time

$$(\Delta v_\perp^2)_{\rm cross} = \frac{N}{R^2}\left(\frac{2Gm}{r_0V}\right)^2 \int_0^R \mathrm{d}b\, b = 2N\left(\frac{Gm}{r_0V}\right)^2. \tag{11}$$

The ratio of the two-body time to the crossing time is thus

$$\frac{t_{\rm 2b}}{t_{\rm cross}} \simeq \frac{v_{\rm typ}^2}{(\Delta v_\perp^2)_{\rm cross}} = \frac{v_{\rm typ}^4 r_0^2 N}{2G^2M^2}, \tag{12}$$

where $M = Nm$ is the system's mass and $v_{\rm typ}$ a typical stellar velocity. Finally we use equation (7) and the fundamental MOND equation $v_{\rm typ}^4 = GMa_0$ to eliminate r_0 and $v_{\rm typ}$ from this equation, and find

$$t_{\rm 2b} = t_{\rm cross}. \tag{13}$$

Thus this naive calculation implies that in the deep-MOND regime the two-body relaxation time is comparable to the crossing time regardless of the number of stars N. The physical origin of this result is clear: equation (9) states that in the deep-MOND regime distant encounters produce much larger deflections than in the Newtonian case, and even in the latter case distant encounters make a large contribution to the relaxation rate.

3 RELAXATION IN A UNIFORM FIELD

The derivation of t_{2b} that we gave in the last section is objectionable on two grounds. First, in MOND all two-body orbits are bound because the potential is asymptotically logarithmic. So our use of the impulse approximation is highly suspect. Second, the derivation assumes that the effects of encounters can simply be added. The dominant encounters are those at impact parameters comparable to the half-mass radius. Each such encounter lasts of order a crossing time, so the many encounters that contribute to $(\Delta v_\perp^2)_{\rm cross}$ in equation (11) occur simultaneously. In the deep-MOND regime the field equation (2) is highly non-linear, and it is far from clear that the effects of different encounters can be simply added. In this section we derive t_{2b} by a different approach that is not open to this objection.

The underlying physical idea is that two-body relaxation is driven by fluctuations in the gravitational potential of a system that is in virial equilibrium. The fluctuations are generated by Poisson noise. We obtain t_{2b} by decomposing the fluctuations into different spatial frequencies, and summing over frequencies rather than over encounters. We shall find that with this approach, the calculation of t_{2b} for MOND differs very little from the corresponding Newtonian calculation. Hence we now rederive the Newtonian relaxation rate with the new formalism, to demonstrate that it produces the familiar result, and to pave the way for the calculation of t_{2b} in MOND.

3.1 Newtonian relaxation

We wish to consider the case in which the underlying system generates a uniform (Newtonian) gravitational field $\mathbf{g}$. Such a field is generated by an infinite sheet of constant density. So we consider the effect that density fluctuations in this sheet have on a star that is located distance z from the sheet. Let $\mathbf{x}$ be a two-dimensional vector of coordinates in the plane of the sheet. Then at $z \neq 0$ Laplace's equation has solutions (e.g. BT §5.3.1)

$$\Phi(\mathbf{x}) = \int \mathrm{d}^2\check{\mathbf{k}}\, \widetilde{\Phi}(\mathbf{k}) \mathrm{e}^{-k|z|} \exp(\imath \mathbf{k}.\mathbf{x}), \tag{14}$$

where $\widetilde{\Phi}$ is an arbitrary function, $k = |\mathbf{k}|$, and $\mathrm{d}^2\check{\mathbf{k}} \equiv \mathrm{d}^2\mathbf{k}/(2\pi)^2$. If the surface density of the sheet is

$$\Sigma(\mathbf{x}) = \int \mathrm{d}^2\check{\mathbf{k}}\, \widetilde{\Sigma}(\mathbf{k}) \exp(\imath \mathbf{k}.\mathbf{x}), \tag{15}$$

then an application of Gauss's theorem shows that

$$\widetilde{\Phi}(\mathbf{k}) = -2\pi G \frac{\widetilde{\Sigma}(\mathbf{k})}{k}. \tag{16}$$

Differentiating (14) with respect to x and integrating with respect to time, we calculate a component of velocity parallel to the sheet that the fluctuating density induces in our test particle:

$$\begin{aligned} v_x(\tau) &= -\int_0^\tau \mathrm{d}t\, \frac{\partial \Phi}{\partial x} \\ &= 2\pi G \int_0^\tau \mathrm{d}t \int \mathrm{d}^2\check{\mathbf{k}}\, \frac{\imath k_x}{k} \widetilde{\Sigma}(\mathbf{k}, t) \mathrm{e}^{-k|z|} \exp(\imath \mathbf{k}.\mathbf{x}). \end{aligned} \tag{17}$$

Squaring v_x and taking an ensemble average, we have

$$\begin{aligned} \left\langle v_x^2(\tau) \right\rangle = -(2\pi G)^2 \int \mathrm{d}^2\check{\mathbf{k}} \int \mathrm{d}^2\check{\mathbf{k}}' \mathrm{e}^{-(k+k')|z|} \frac{k_x k_x'}{kk'} \\ \times \int \mathrm{d}t \int \mathrm{d}t' \exp[\imath(\mathbf{k}+\mathbf{k}').\mathbf{x}] \left\langle \widetilde{\Sigma}(\mathbf{k}, t) \widetilde{\Sigma}(\mathbf{k}', t') \right\rangle. \end{aligned} \tag{18}$$

Let $\mathbf{x}_\alpha$ denote the location within the sheet of star α. Then

$$\Sigma(\mathbf{x}, t) = m \sum_\alpha \delta[\mathbf{x}_\alpha(t) - \mathbf{x}] \tag{19}$$

so

$$\widetilde{\Sigma}(\mathbf{k}, t) = m \sum_\alpha \exp[-\imath \mathbf{k}.\mathbf{x}_\alpha(t)]. \tag{20}$$

For the moment we assume that the velocity $\mathbf{v}_\alpha$ of star α lies within the sheet. Then to a sufficient approximation $\mathbf{x}_\alpha(t') = \mathbf{x}_\alpha(t) + (t' - t)\mathbf{v}_\alpha$ and we have for uncorrelated stars

$$\begin{aligned} \left\langle \widetilde{\Sigma}(\mathbf{k}, t) \widetilde{\Sigma}(\mathbf{k}', t') \right\rangle = m^2 \\ \times \left\langle \sum_\alpha \exp[-\imath \mathbf{x}_\alpha.(\mathbf{k}+\mathbf{k}')] \exp[\imath(t'-t)\mathbf{k}'.\mathbf{v}_\alpha] \right\rangle. \end{aligned} \tag{21}$$

Since the velocities of stars are independent of their positions, the ensemble average above can be expressed as the product of two ensemble averages, one over $\mathbf{x}_\alpha$ and the other over $\mathbf{v}_\alpha$. With $\mathbf{K} = \mathbf{k} + \mathbf{k}'$ we have

$$\langle \exp(-\imath \mathbf{x}_\alpha.\mathbf{K}) \rangle = \int_A \frac{\mathrm{d}^2\mathbf{x}}{A} \exp(-\imath \mathbf{x}.\mathbf{K}) = \frac{(2\pi)^2}{A} \delta(\mathbf{K}), \tag{22}$$

where A is any large area of the sheet, so

$$\sum_\alpha \langle \exp(-\imath \mathbf{x}_\alpha.\mathbf{K}) \rangle = (2\pi)^2 n \delta(\mathbf{K}), \tag{23}$$

where n is the number of stars per unit area. For the other ensemble average we have for a Maxwellian distribution of velocities

$$\begin{aligned} \left\langle \exp[\imath(t'-t)\mathbf{k}'.\mathbf{v}_\alpha] \right\rangle &= \int \frac{\mathrm{d}^2\mathbf{v}}{2\pi\sigma^2} \mathrm{e}^{-v^2/2\sigma^2} \exp[\imath(t'-t)\mathbf{k}'.\mathbf{v}] \\ &= \exp\left[-\tfrac{1}{2}(t'-t)^2 k'^2 \sigma^2\right]. \end{aligned} \tag{24}$$

Substituting equations (21), (23) and (24) into equation (18) and integrating over t' and t, we find for $\tau \gg k\sigma$

$$\begin{aligned} \frac{\left\langle v_x^2(\tau) \right\rangle}{\tau} &= \frac{(2\pi)^{5/2} G^2 m^2 n}{\sigma} \int \mathrm{d}^2\check{\mathbf{k}}\, \mathrm{e}^{-2k|z|} \frac{k_x^2}{k^3} \\ &= \frac{(2\pi)^{3/2} G^2 m^2 n}{4|z|\sigma}. \end{aligned} \tag{25}$$

From equation (25) we can recover the standard expression for the Newtonian diffusion coefficient by summing the contributions from many sheets. If ρ is the (homogeneous)

mass density due to stars, then $\rho \mathrm{d}z = nm$, and the overall diffusion coefficient is

$$\frac{\left\langle v_x^2(\tau)\right\rangle_z}{\tau} = \frac{(2\pi)^{3/2}G^2\rho m}{2\sigma}\int_0^\infty \frac{\mathrm{d}z}{z} = \frac{(2\pi)^{3/2}G^2\rho m}{2\sigma}\ln\Lambda, \quad (26)$$

with $\Lambda = z_{\max}/z_{\min}$. This diffusion coefficient may be compared with half the value of $D(\Delta v_\perp^2)$ in equation (8-68) of BT. Taking the limit $X \to 0$ of small test-particle velocities we find

$$\frac{2\left\langle v_x^2(\tau)\right\rangle_z}{\tau D(\Delta v_\perp^2)} = \frac{6\pi}{8}. \quad (27)$$

Thus this derivation agrees with the classical one to within the uncertainties inherent in either approach. The weakest part of the present derivation is the assumption that the velocities of field stars lie within planes $z =$ constant. Relaxing this assumption would reduce the auto-correlation of each sheet's surface density below that given by equation (21) and introduce correlations between the densities of different sheets. When the change in z during an encounter is small compared to the distance of the particle from the point of observation, little will have changed physically from the case of constant z, so the new correlations will almost exactly compensate for the lost contribution to the autocorrelation. This argument suggests that the error introduced by confining particles to planes of constant z is not large, as is also indicated by the agreement between our value of the diffusion coefficient and that obtained in the standard way.

3.2 Relaxation in MOND

So long as the total number of stars in the system is large, and we avoid special points of symmetry such as the system's centre, the fluctuations in the gravitational field are small compared to the MOND mean field $\mathbf{g}$. Hence we may linearize the field equation (2) around $\mathbf{g}$ and solve a linear field equation for the component of the potential that drives relaxation. As in the Newtonian case, we decompose the perturbing density field into sheets normal to the mean gravitational field. We find the potential fluctuations due to each sheet, and add the effects of the sheets.

M86 shows that when the field equation (2) is perturbed about a uniform gravitational field $\mathbf{g}_0$, the first-order perturbations to the potential and density are connected by

$$\left(\nabla^2 + \frac{\partial^2}{\partial z^2}\right)\Phi = \frac{a_0}{g_0}4\pi G\rho, \quad (28)$$

where ∇^2 is the full three-dimensional Laplacian operator. Thus the solutions (14) to Laplace's equation must be replaced by

$$\Phi(\mathbf{x}) = \int \mathrm{d}^2\check{\mathbf{k}}\,\widetilde{\Phi}(\mathbf{k})\mathrm{e}^{-k|z|/\sqrt{2}}\exp(\imath\mathbf{k}.\mathbf{x}), \quad (29)$$

and equation (16) becomes

$$\widetilde{\Phi}(\mathbf{k}) = -\frac{a_0}{\sqrt{2}g_0}2\pi G\frac{\widetilde{\Sigma}(\mathbf{k})}{k}. \quad (30)$$

The Newtonian calculation carries over with two substitutions: $|z| \to |z|/\sqrt{2}$ and $m \to (a_0/\sqrt{2}g_0)m$. Equation (25) for the contribution of a single sheet to the diffusion coefficient becomes

$$\frac{\left\langle v_x^2(\tau)\right\rangle}{\tau} = \frac{(2\pi)^{3/2}G^2m^2n}{4|z|\sigma}\frac{a_0^2}{\sqrt{2}g_0^2}, \quad (31)$$

so the diffusion coefficient is larger than in the Newtonian case by a factor $a_0^2/(\sqrt{2}g_0^2)$.

Consider now two systems that contain identical distributions of stars in phase space, but differ in the way gravity works: in one system gravity is Newtonian, while in the other it is described by MOND. In the Newtonian case we augment the gravitational field of the stars with a fixed background field to ensure overall dynamical equilibrium. Then the two-body relaxation times in the two systems are in the inverse ratio of their diffusion coefficients, which from equations (25) and (31) is

$$\frac{t_{2\mathrm{b}}^{\mathrm{M}}}{t_{2\mathrm{b}}^{\mathrm{N}}} = \frac{\sqrt{2}g_0^2}{a_0^2}. \quad (32)$$

This result will remain true when the fixed background field in the Newtonian system is replaced by the field generated by a distribution of DM particles provided individual DM particles are much lighter than stars – the particles then make negligible contributions to the fluctuations in the overall gravitational acceleration[1].

In the deep-MOND regime, equation (5) holds, so eliminating a_0 from equation (32), the ratio of two-body times becomes

$$\frac{t_{2\mathrm{b}}^{\mathrm{M}}}{t_{2\mathrm{b}}^{\mathrm{N}}} = \frac{\sqrt{2}g_{\mathrm{N}}^2}{g_0^2} = \frac{\sqrt{2}}{(1+\mathcal{R})^2}, \quad (33)$$

where

$$\mathcal{R} \equiv \frac{M_{\mathrm{DM}}}{M_*} \quad (34)$$

is the ratio of the DM to stellar mass in the Newtonian system.

3.3 Dynamical friction

Equation (33) has important implications for the magnitude of dynamical friction in MOND because the dynamical friction experienced by a body as it moves through a population of background objects that are in thermal equilibrium, is proportional to its diffusion coefficient in velocity space. This relation can be verified for coefficients calculated under the assumption of standard Newtonian gravity, but, as Chandrasekhar (1943) pointed out, it follows from the general principles of statistical mechanics. This fact ensures that the relation is valid regardless of the law of gravity.

In the local approximation, the Fokker-Planck equation for the evolution of the distribution function f of a population of 'test' objects can be written

$$\frac{\mathrm{d}f}{\mathrm{d}t} = -\frac{\partial}{\partial\mathbf{v}}\cdot\mathbf{S}, \quad (35)$$

where the flux $\mathbf{S}$ of stars in velocity space is [BT eq. (8-57)]

$$S_i = -fD_i + \tfrac{1}{2}\sum_j \frac{\partial}{\partial v_j}(fD_{ij}). \quad (36)$$

[1] In a Newtonian system with field particles of two different species, with masses m_* and m_{DM} and densities ρ_* and ρ_{DM}, the two body relaxation time of a test particle of mass m_{a} is $t_{2\mathrm{b}}^{\mathrm{N}} \propto 1/(m_*\rho_* + m_{\mathrm{DM}}\rho_{\mathrm{DM}})$.

Here $D_i = D(\Delta v_i)$ and $D_{ij} = D(\Delta v_i\,\Delta v_j)$ are the usual first- and second-order diffusion coefficients. Let the scattering objects have mass $m_{\rm f}$ and be in thermal equilibrium at inverse temperature $\beta = (m_{\rm f}\sigma^2)^{-1}$. Then the principle of detailed balance implies that $\mathbf{S}$ will vanish when $f \sim \exp(-\beta H)$, where $H = m_{\rm a}(\frac{1}{2}v^2 + \Phi)$ is the Hamiltonian that governs the motion of the test objects. In a frame in which D_{ij} is diagonal it then follows that

$$\begin{aligned} D_i &= -\tfrac{1}{2}\beta m_{\rm a} v_i D_{ii} + \tfrac{1}{2}\frac{\partial}{\partial v_i} D_{ii} \\ &= -\left(\frac{m_{\rm a}}{2m_{\rm f}} - \sigma^2 \frac{\partial}{\partial v_i^2}\ln D_{ii}\right)\frac{D_{ii}}{\sigma^2} v_i \end{aligned} \tag{37}$$

(no sum over repeated indices). For $v_i \lesssim \sigma$ the term with the logarithmic derivative is of order unity – in the case of a Maxwellian distribution of scatterers it evaluates to $-3/10$ at $v_i = 0$ [BT, eq. (8-65)]. Hence for $m_{\rm a} \gtrsim 4m_{\rm f}$ equation (37) implies that v_i decays exponentially in a characteristic time $t_{\rm fric}$ that is related to the two-body relaxation time by

$$\frac{t_{\rm fric}}{t_{\rm 2b}} = \frac{2m_{\rm f}}{m_{\rm a}}; \tag{38}$$

this result is consistent with the Spitzer (1987) relation $t_{\rm fric}/t_{\rm 2b} = 2m_{\rm f}/(m_{\rm f} + m_{\rm a})$, when $m_{\rm a} \gtrsim 4m_{\rm f}$. From this result and eq. (33) it follows that in MOND the friction time is reduced by a factor $(1+\mathcal{R})^2/\sqrt{2}$ over the value it would have in a Newtonian system with the same stellar mass and a *fixed* auxiliary gravitational field.

Whereas the diffusion coefficients were unchanged when the fixed gravitational field was replaced by the field of swarms of low-mass DM particles, this replacement enhances dynamical friction by a factor $(1+\mathcal{R})$. Hence[2].

$$\frac{t^{\rm N}_{\rm fric}}{t^{\rm N}_{\rm 2b}} = \frac{2m_{\rm f}}{m_{\rm a}}(1+\mathcal{R})^{-1}, \tag{39}$$

and

$$\frac{t^{\rm M}_{\rm fric}}{t^{\rm N}_{\rm fric}} = \frac{\sqrt{2}}{1+\mathcal{R}}. \tag{40}$$

4 EXTENSION TO SPHERICAL SYSTEMS

Given that stellar systems are frequently approximately spherical, and rarely have the plane-parallel symmetry that we assumed in the last section, we try to adapt the preceding calculation to a spherical system. The appendix derives an expression for the Newtonian diffusion coefficient experienced by a star that is stationary near the centre of a spherical system.

If we are to use perturbation theory to carry this Newtonian analysis over to the deep-MOND regime, the unperturbed acceleration g_0 should be non-vanishing and less than a_0 at the location of the test star. For simplicity the system's density profile is taken to be a power law in radius $\rho \sim r^{-\gamma}$, and the requirement that at all radii g_0/a_0 be of order but less than unity then implies $\gamma = 1$. We show that in this case the acceleration caused by the fluctuations becomes comparable to g_0 as one approaches the centre, and the linearized field equation ceases to be valid. We infer from this result that in the deep-MOND regime the centres of all systems must be homogeneous, since non-linear fluctuations in the acceleration will soon disrupt the cusp in the density profile that generates the assumed constant acceleration g_0. Unfortunately, perturbation theory cannot be used to calculate the central relaxation time of a system with a constant-density core.

Thus this analysis does not lead to a value of $t_{\rm 2b}$ for systems in the deep-MOND regime, but it does suggest that at the centres of these systems two-body relaxation is very much more rapid than in the Newtonian case.

5 ASTROPHYSICAL APPLICATIONS

We now apply the results of Section 3 to stellar systems that in Newtonian dynamics would turn out to be DM dominated, namely to dwarf galaxies, to galaxy clusters and to galaxy groups: all these systems are in the deep-MOND regime.

Equation (8-71) of BT gives the Newtonian two-body relaxation time for a system with a given velocity dispersion σ and density of scatterers ρ_*. As in BT, we substitute into this equation values of σ and ρ_* appropriate to the system's half-mass radius. We obtain a result for the reference relaxation time that is analogous to equation (8-72) of BT but different because now $\sigma^2 \simeq 0.4GM_*(1+\mathcal{R})/r_{\rm h}$ on account of the presence of DM. With $\Lambda = 0.4N(1+\mathcal{R})$ we have

$$t^{\rm N}_{\rm rh} \simeq 0.66\,{\rm Gyr}\frac{(1+\mathcal{R})^{3/2}}{\ln\Lambda}\frac{M_\odot}{m}\left(\frac{r_{\rm h}}{\rm pc}\right)^{3/2}\left(\frac{M_*}{10^5 M_\odot}\right)^{1/2}, \tag{41}$$

while equation (33) now implies that the relaxation time for MOND is

$$t^{\rm M}_{\rm rh} \simeq 0.9\,{\rm Gyr}\frac{(1+\mathcal{R})^{-1/2}}{\ln\Lambda}\frac{M_\odot}{m}\left(\frac{r_{\rm h}}{\rm pc}\right)^{3/2}\left(\frac{M_*}{10^5 M_\odot}\right)^{1/2}. \tag{42}$$

5.1 Dwarf galaxies

For a dwarf galaxy such as Draco, $M_* \simeq 2.6\,10^5 M_\odot$, $r_{\rm h} \simeq 200h^{-1}$ pc, and $\mathcal{R} \simeq 100$ (see. e.g., Mateo 1998, Kleyna et al. 2002), so the reduction factor is enormous ($\sim 10^4$). However, $t^{\rm N}_{\rm rh} \simeq 10^5$ Gyr, so $t^{\rm M}_{\rm rh}$ is still is slightly longer than the Hubble time.

Since in MOND the dynamical-friction time for inspiralling of an object of mass $m_{\rm a}$ is shorter than the two-body time by $\sim M_\odot/m_{\rm a}$, any object significantly more massive than a star will have spiralled to the centre of a dwarf galaxy. In particular, any black holes with masses $\gtrsim 10 M_\odot$ should have collected at the centre. Since globular clusters have masses $\gtrsim 10^4 M_\odot$, they are liable to spiral to the galaxy centre even in Newtonian dynamics. In MOND they spiral in on essentially a dynamical time. Consequently, the possession of globular clusters by a dwarf galaxy that is in the deep-MOND regime would be problematic for MOND. Interestingly, the dwarfs with large values of $\mathcal{R}$, namely Draco,

[2] A system with comparable masses of luminous and DM violates equation (38) because the stars and DM particles are not in thermal equilibrium with one another. The standard treatment of dynamical friction shows that for a test particle of mass $m_{\rm a}$ $t^{\rm N}_{\rm fric} \propto 1/[(m_{\rm a} + m_*)\rho_* + (m_{\rm a} + m_{\rm DM})\rho_{\rm DM}]$, and from Footnote 1, when $m_{\rm a} \gtrsim$ of m_* and $m_{\rm DM}$, and $m_* \gtrsim m_{\rm DM}$ equation (35) is reobtained.

UMi, Carina, and Sextans, have no globular clusters listed in Table 8 of Mateo (1998).

5.2 Galaxy clusters

If $M_{\rm T}$ is the total cluster mass, we have $M_* \simeq 0.05 M_{\rm T}$, $M_{\rm gas} \simeq 0.15 M_{\rm T}$, $M_{\rm DM} \simeq 0.8 M_{\rm T}$ (see, e.g., Böhringer 1996), so $\mathcal{R} \simeq 4$. Adopting $N \simeq 100$, $M_* \simeq 1.7\,10^{13} M_\odot$, $r_{\rm h} \simeq 580\,{\rm kpc}$, and $\sigma = 1000\,{\rm kms}^{-1}$, we find $t_{\rm rh}^{\rm N} \simeq 48\,{\rm Gyr}$ and $t_{\rm rh}^{\rm M} \simeq 2.7\,{\rm Gyr}$.

In MOND dynamical friction will cause a galaxy of mass m_a to spiral to the cluster centre on a timescale $\sim 5(\overline{m}/m_a)\,{\rm Gyr}$, where $\overline{m}$ is the mass-weighted mean galactic mass. It follows that in MOND luminosity segregation should be well advanced within galaxy clusters.

5.3 Galaxy groups

For a typical galaxy group, $N \simeq 5$, $M_* \simeq 5.2\,10^{11} M_\odot$, $\mathcal{R} = 16$, $\sigma \simeq 200\,{\rm kms}^{-1}$, and $r_{\rm h} \simeq 380\,{\rm kpc}$. Hence $t_{\rm rh}^{\rm N} \simeq 70\,{\rm Gyr}$ and $t_{\rm rh}^{\rm M} \simeq 340\,{\rm Myr}$. If one (improperly!) applied the concepts of Newtonian stellar dynamics, one would conclude that in MOND galaxy groups should have already evaporated because in the Newtonian case the evaporation time is $\sim 10 t_{\rm rh}$. The logarithmic asymptotic form of MOND's two-body potential implies that it is impossible to escape from a perfectly isolated MOND system[3]. However, the shortness of $t_{\rm rh}^{\rm M}$ for galaxy groups suggests that relaxation will have significantly increased the central densities of groups. Luminosity segregation will proceed on a dynamical timescale.

6 DISCUSSION AND CONCLUSION

A naive extension to MOND of the standard derivation of the two-body relaxation time $t_{\rm 2b}$ leads to the conclusion that $t_{\rm 2b}$ is comparable to the crossing time for any value of the number N of stars in the system. The derivation is open to objection on at least two grounds, so we have developed an approach to the calculation of the Newtonian relaxation time that can be straightforwardly adapted to MOND. This focuses on the effect of density fluctuations, which must be small in the limit of large N, and calculates the corresponding potential fluctuations by linearizing MOND's field equation around a uniform background field $g_0 \ll a_0$. This analysis reveals that in MOND $t_{\rm 2b}$ scales with N in the same way as it does in the Newtonian case, but is smaller by a factor $\sim (1+\mathcal{R})^2$, where $\mathcal{R}$ is the ratio of the apparent DM and stellar masses.

We show that the timescale $t_{\rm fric}$ on which dynamical friction causes an object of mass $m_{\rm a}$ to spiral in through a population of scatterers of mass $m_{\rm f}$ is $t_{\rm fric}^{\rm M} = (2m_{\rm f}/m_{\rm a}) t_{\rm 2b}^{\rm M}$ in the case of MOND, while the Newtonian time is $t_{\rm fric}^{\rm N} \simeq (1+\mathcal{R}) t_{\rm fric}^{\rm M}$, provided individual DM particles have negligible masses.

Application of these results to DM-dominated systems shows that at the half-mass radius, the two-body relaxation times for MOND of both dwarf galaxies and clusters of galaxies are of order the Hubble time. Consequently, MOND predicts that objects in these systems that have masses more than ~ 10 times the mean mass will have spiralled to the systems' centres. In particular, any black holes formed in core-collapse supernovae should have collected at the centres of dwarf galaxies, and luminosity segregation should be far advanced in clusters of galaxies. The globular clusters of dwarf galaxies should spiral inwards on a dynamical timescale, so the existence of globular clusters in DM-dominated dwarf galaxies would be problematic for MOND.

In MOND the two-body and dynamical-friction timescales of groups of galaxies are only $\approx 300\,{\rm Myr}$. A Newtonian system with such a short relaxation time would have evaporated completely, but evaporation is probably impossible in MOND. However, in MOND relaxation will surely have significantly modified the density profiles of galaxy groups, and induced substantial luminosity segregation. Further work is required to determine whether such evolution is compatible with observation.

The modification of Poisson equation given by equation (2) is only one possible formulation fo MOND. We have not considered relaxation in modified inertia theories, which give very similar predictions for rotation curves and global mass discrepancies (Milgrom 2002). These theories could yield very different predictions for relaxation phenomena.

[3] We note that it is generally supposed that the galaxy disks are made of shattered star clusters and associations: while tidal destruction will still work, the failure of evaporation in outlying parts of galaxies might lead to the survival of surprising numbers of low-mass star clusters.

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APPENDIX A: RELAXATION AT THE CENTRE OF A SPHERICAL SYSTEM

A1 Newtonian case

We consider the stochastic acceleration of a particle that is initially stationary at the centre of a spherical system. We expand the system's gravitational potential, generated by the (random) density field

$$\rho_1 = \sum_\alpha m_\alpha \delta(\mathbf{x} - \mathbf{x}_\alpha), \tag{A1}$$

in spherical harmonics. At the origin, only the dipole term makes a non-vanishing contribution to the force on a particle. Moreover, near the origin the force contributed by the lth multipole varies as r^{l-1} times a factor that changes sign each time the particle passes the origin for even l. Hence averaged along a trajectory through the orgin, the force from multipoles with $l > 1$ is smaller by that from the dipole term by a factor that vanishes with the square of the apocentric radius. We therefore concentrate on the dipole term, which is (BT §2.4)

$$\Phi_1(r,\theta,\phi) = -\tfrac{4}{3}\pi G \sum_m Y_1^m(\theta,\phi) r \int_r^\infty \mathrm{d}a\, \rho_{1m}(a), \tag{A2}$$

where

$$\begin{aligned}\rho_{1m}(a) &= \int_{4\pi} \mathrm{d}^2\Omega\, Y_1^{m*} \rho(a,\theta,\phi)\\ &= \frac{1}{a^2}\frac{\mathrm{d}}{\mathrm{d}a}\int_0^a \mathrm{d}r\, r^2 \int_{4\pi} \mathrm{d}^2\Omega\, Y_1^{m*}\rho(r,\theta,\phi)\\ &= \frac{1}{a^2}\frac{\mathrm{d}}{\mathrm{d}a}\int \mathrm{d}^3\mathbf{x}\, Y_1^{m*}\sum_\alpha m_\alpha \delta(\mathbf{x}-\mathbf{x}_\alpha)\\ &= \frac{1}{a^2}\frac{\mathrm{d}}{\mathrm{d}a}\sum_\alpha m_\alpha Y_1^{m*}(\alpha),\end{aligned} \tag{A3}$$

where the sum over α is extended to particles inside the sphere of radius a. Differentiating (A2) with respect a generic direction (which without loss of generality we assume to be z), and evaluating the obtained expression at $r = 0$, we have

$$\dot{v}_z = -\frac{\partial \Phi_1}{\partial z} = \tfrac{4}{3}\pi G \int_0^\infty \mathrm{d}a\, \rho_{10}(a) \tag{A4}$$

because the contributions from $m = \pm 1$ vanish. Integration with respect to time gives

$$v_z(\tau) = \tfrac{4}{3}\pi G \int_0^\tau \mathrm{d}t \int_0^\infty \mathrm{d}a\, \rho_{10}(a). \tag{A5}$$

Squaring and taking the ensemble average we find

$$\left\langle v_z^2 \right\rangle = \frac{(4\pi G)^2}{9} \iint \mathrm{d}a\mathrm{d}a' \iint \mathrm{d}t\mathrm{d}t'\, \left\langle \rho_{10}(a,t)\rho_{10}(a',t')\right\rangle . \tag{A6}$$

The expectation value in the integrand is non-negligible only for $|a - a'|$ and $|t - t'|$ sufficiently small. With (A3) we have in the case that all particles have equal masses

$$\begin{aligned}C(a,t,t') &\equiv \int \mathrm{d}a'\, \left\langle \rho_{10}(a,t)\rho_{10}(a',t')\right\rangle\\ &= \frac{m^2}{a^2}\frac{\mathrm{d}}{\mathrm{d}a}\int \frac{1}{a'^2}\frac{\mathrm{d}}{\mathrm{d}a'}\sum_{\alpha,\beta}\left\langle Y_1^0(\alpha,t)Y_1^0(\beta,t')\right\rangle\\ &= \frac{m^2}{a^2}\frac{\mathrm{d}}{\mathrm{d}a}\int \frac{1}{a'^2}\frac{\mathrm{d}}{\mathrm{d}a'}\sum_{\alpha}\left\langle Y_1^0(\alpha,t)Y_1^0(\alpha,t')\right\rangle,\end{aligned} \tag{A7}$$

where we have assumed that the particles are mutually uncorrelated. If σ is the characteristic tangential velocity of particle α, the expectation value in this equation vanishes after a time of order r_α/σ. Therefore we replace the integral over t' of $C(a,t,t')$ by a/σ times half its peak value. To estimate the latter we observe that by the Monte-Carlo theorem

$$1 = \int \mathrm{d}^2\Omega\, |Y_1^0|^2 = \frac{4\pi}{N}\sum_{\alpha=1}^N |Y_1^0(\alpha)|^2. \tag{A8}$$

Hence

$$\int \mathrm{d}t'\, C(a,t,t') = \frac{m^2}{8\pi a^2}\frac{\mathrm{d}}{\mathrm{d}a}\left(\frac{1}{a^2}\frac{a}{\sigma}N\right). \tag{A9}$$

Inserting this expression into (A6) we have

$$D \equiv \frac{\left\langle v_z^2(\tau)\right\rangle}{\tau} = \frac{2\pi (Gm)^2}{9\sigma}\int \frac{\mathrm{d}(N/a)}{a^2}. \tag{A10}$$

If the stellar density ρ/m is a constant from some smallest radius out to some maximum radius $r_{\max}$ and then zero, then $N = \frac{4}{3}\pi a^3(\rho/m)$ and the diffusion coefficient D becomes

$$D = \frac{16\pi^2 G^2 m\rho}{27\sigma}\ln\left(\frac{r_{\max}}{r_{\min}}\right). \tag{A11}$$

This should be compared with half the value of $\lim_{X\to 0} D(\Delta v_\perp^2)$ in eq. (8-68) of BT. One finds

$$\frac{2D}{\lim_{X\to 0} D(\Delta v_\perp^2)} = \frac{(2\pi)^{3/2}}{9} = 1.75\ . \tag{A12}$$

As in the case of plane-parallel geometry we obtain a diffusion coefficient that is larger by a factor $\lesssim 2$ than its conventional value.

A2 Case of the deep-MOND regime

For the reasons given in the main text we assume that the system's density profile obeys the power law $\rho \sim 1/r$, which yields a constant radial acceleration $\mathbf{g_0} = -\sqrt{a_0 G M_0/r_0^2}\,\mathbf{e}_r$, where M_0 is the mass interior to the fiducial radius r_0.

The perturbed potential Φ satisfies eq. (7) of M86 with $\mu_0 = g_0/a_0$ and $L_0 = 1$ because we are in the deep-MOND regime. Hence

$$\frac{2}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right) - \frac{\mathcal{L}^2}{r^2}\Phi = \frac{g_0}{a_0}4\pi G\rho \tag{A13}$$

where $\mathcal{L}$ is the angular part of the Laplacian. We seek the dipole term in the expansion of Φ in spherical harmonics

$$\Phi(r,\vartheta,\varphi) = \sum_{l=0}^\infty \sum_{m=-l}^l \Phi_{lm}(r) Y_l^m(\vartheta,\varphi), \tag{A14}$$

which satisfies

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi_{10}}{\partial r}\right) - \frac{\Phi_{10}}{r^2} = \frac{g_0}{a_0}2\pi G\rho_{10}. \tag{A15}$$

The Green's function $u(r,r)$ for this equation satisfies

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) - \frac{u}{r^2} = \frac{g_0}{a_0}2\pi G\delta(r,r'). \tag{A16}$$

From the power-law solutions to the homogeneous equation it follows that

$$u(r,r') = \begin{cases} A(r')r^{\alpha-1/2} & \text{for } r < r' \\ B(r')r^{-\alpha-1/2} & \text{for } r > r', \end{cases} \tag{A17}$$

where $A(r)$ and $B(r)$ are functions to be determined from the rhs of eq. (A16) and $\alpha = \sqrt{5}/2$. Consequently, near the origin the dipole term varies as $r^{(\sqrt{5}-1)/2} \sim r^{0.62}$ and

the acceleration to which it gives rise diverges as $r^{-0.38}$. Hence no matter how small the fluctuations in ρ are, the linearization of (2) breaks down sufficiently near the origin. We conclude that sufficiently near the origin the fluctuating part of the field is comparable to, or larger than, g_0. In these circumstances the fluctuations will soon disrupt the cusp in the density distribution that generates g_0. It seems therefore that in the deep-MOND regime the centres of all systems must be homogeneous. Unfortunately, perturbation theory cannot be used to calculate the central relaxation time of such a system.